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The configuration space of 6 points in $\mathbf{P}^2$, the moduli space of cubic surfaces and the Weyl group of type E_6

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1. Introduction

My first plan of the talk is to explain my study on the hypergeometric system $E(3,6)$ of type (3,6) ([8]). The system in question admits Σ_6-action, where Σ_6 is the symmetric group on 6 letters. This follows from that $E(3,6)$ lives in the configurations space $\mathbf{P}_2^6$ of 6 points in $\mathbf{P}^2$ which admits Σ_6-action as permutations of the 6 points. Recently M. Yoshida (Kyushu Univ.) pointed out that the Σ_6-action on the space $\mathbf{P}_2^6$ is naturally extended to $W(E_6)$-action, where $W(E_6)$ is the Weyl group of type E_6 (cf. [3]). Moreover, he told me that B. Hunt studied relations between the $W(E_6)$-action in question and the $W(E_6)$-invariant quintic hypersurface of $\mathbf{P}^5$.

Reading his note [4], I felt that it is an interesting exercise for REDUCE user to show whether his conjecture is true or not. For this reason, I changed the original plan and I restrict my attention to the study on $W(E_6)$-actions on $\mathbf{P}^5$ and on $\mathbf{P}_2^6$, namely, to the birational geometry related with the hypergeometric system $E(3,6)$.

It is better for the readers who are interested in SYMBOLIC COMPUTATION to read section 6 first.

2. The hypergeometric function of type (3,6)

Though I don't treat it in this note, I begin this note with introducing the hypergeometric function of type (3,6):

$$E(a_0,a_2,a_3,a_5,a_6;x_1,x_2,y_1,y_2)=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}A_{m_1,m_2,n_1,n_2}x_1^{m_1}x_2^{m_2}y_1^{n_1}y_2^{n_2}$$

where

$$A_{m_1,m_2,n_1,n_2}=\frac{(a_2,m_1+m_2)(a_3,n_1+n_2)(1-a_5,m_1+n_1)(1-a_6,m_2+n_2)}{m_1!m_2!n_1!n_2!(a_0,m_1+m_2+n_1+n_2)}.$$

By definition, $E(a_0,a_2,a_3,a_5,a_6;x_1,x_2,y_1,y_2)$ has parameters a_j $(j=0,2,3,5,6)$. This function is one of 4 variables generalizations of Gaussian hypergeometric function. It is known that the singularities of the system of differential equations whose solution is $E(a_0, a_2, a_3, a_5, a_6; x_1, x_2, y_1, y_2)$ is contained in the union of the 14 hypersurfaces $T_j : p_j = 0\ (1 \le j \le 14)$, where

$$p_1 = x_1y_2 - x_2y_1 - x_1 + x_2 + y_1 - y_2, \quad p_2 = y_1 - 1, \quad p_3 = x_1 - 1,$$
$$p_4 = y_2 - 1, \quad p_5 = x_2 - 1, \quad p_6 = y_1 - y_2, \quad p_7 = x_1 - x_2, \quad p_8 = x_1 - y_1,$$
$$p_9 = x_2 - y_2, \quad p_{10} = x_1y_2 - x_2y_1, \quad p_{11} = x_2, \quad p_{12} = x_1, \quad p_{13} = y_2, \quad p_{14} = y_1.$$

We define birational transformations $s_j\ (1 \le j \le 5)$ on $\mathbf{C}^4$ by

$$s_1 : (x_1, x_2, y_1, y_2) \longrightarrow (\frac{1}{x_1}, \frac{1}{x_2}, \frac{y_1}{x_1}, \frac{y_2}{x_2}),$$
$$s_2 : (x_1, x_2, y_1, y_2) \longrightarrow (y_1, y_2, x_1, x_2),$$
$$s_3 : (x_1, x_2, y_1, y_2) \longrightarrow (\frac{x_1 - y_1}{1 - y_1}, \frac{x_2 - y_2}{1 - y_2}, \frac{y_1}{y_1 - 1}, \frac{y_2}{y_2 - 1}),$$
$$s_4 : (x_1, x_2, y_1, y_2) \longrightarrow (\frac{1}{x_1}, \frac{x_2}{x_1}, \frac{1}{y_1}, \frac{y_2}{y_1}),$$
$$s_5 : (x_1, x_2, y_1, y_2) \longrightarrow (x_2, x_1, y_2, y_1).$$

Then the group generated by $s_j\ (1 \le j \le 5)$ is identified with Σ_6 because

$$s_j^2 = id.\ (1 \le j \le 5), \quad s_js_k = s_ks_j\ (|j - k| > 1),$$
$$s_js_ks_j = s_ks_js_k\ (|j - k| = 1).$$

Let r be a birational transformation on $\mathbf{C}^4$ defined by

$$r : (x_1, x_2, y_1, y_2) \longrightarrow (1/x_1, 1/x_2, 1/y_1, 1/y_2).$$

Then the group $\tilde{G}$ generated by $s_1, \cdots, s_5$ and r is isomorphic to the Weyl group $W(E_6)$ of type E_6 which will be seen later (cf. [3], [4]).

We define the hypersurface $T_{15} : p_{15} = 0$, where

$$p_{15} = x_1y_2(1 - y_1)(1 - x_2) - x_2y_1(1 - x_1)(1 - y_2).$$

It follows from the definition that $s_1, \cdots, s_5, r$ and therefore all the elements of $\tilde{G}$ are biregular outside the union T of the hypersurfaces $T_j\ (1 \le j \le 15)$.

3. The Weyl group $W(E_6)$

Let $E_{\mathbf{R}}$ be a Cartan subalgebra of a compact Lie algebra of type E_6, i.e. $E_{\mathbf{R}} \simeq \mathbf{R}^6$. Let $t = (t_1, t_2, t_3, t_4, t_5, t_6)$ be a coordinate system of $E_{\mathbf{R}}$ such that the roots of type E_6 are:

$$\pm(t_i \pm t_j), \quad 1 \le i < j \le 5$$
$$\pm\frac{1}{2}(\delta_1t_1 + \delta_2t_2 + \delta_3t_3 + \delta_4t_4 + \delta_5t_5 + \delta_6t_6)$$

(where $\delta_j = \pm 1$ and $\prod_j \delta_j = 1$). Note that compared with the notation in [1], our variables $t_i = \epsilon_i$, $i = 1, \cdots, 5$, while our coordinate t_6 is denoted $\epsilon_6 - \epsilon_7 - \epsilon_8$ in [1]. We now introduce the following linear forms on $E_{\mathbf{R}}$:

$$h = -\frac{1}{2}(t_1 + \cdots + t_6),$$

$$h_{1j} = -t_{j-1} + h_0, \quad j = 2, \cdots, 6$$

$$h_{jk} = t_{j-1} - t_{k-1}, \quad j, k \neq 1$$

$$h_{1jk} = -t_{j-1} - t_{k-1}, \quad j, k \neq 1$$

$$h_{jkl} = -t_{j-1} - t_{k-1} - t_{l-1} + h_0, \quad j, k, l \neq 1$$

where

$$h_0 = \frac{1}{2}(t_1 + \cdots + t_5 - t_6).$$

Then the totality of h, h_{ij}, h_{ijk} forms a set of positive roots of type E_6. Let s (resp. s_{ij}, s_{ijk}) be the reflection on $E_{\mathbf{R}}$ with respect to the hyperplane $h = 0$ (resp. $h_{ij} = 0, h_{ijk} = 0$). Then the Weyl group of type E_6 which is denoted by $W(E_6)$ in this note is the group generated by the 36 reflections defined above.

As a system of simple roots, we take

$$\alpha_1 = h_{12}, \quad \alpha_2 = h_{123}, \quad \alpha_3 = h_{23}, \quad \alpha_4 = h_{34}, \quad \alpha_5 = h_{45}, \quad \alpha_6 = h_{56}.$$

Then the Dynkin diagram is:

$$\begin{array}{ccccccccc} \alpha_1 & — & \alpha_3 & — & \alpha_4 & — & \alpha_5 & — & \alpha_6 \\ & & & & | & & & & \\ & & & & \alpha_2 & & & & \end{array}$$

Let g_j be the reflection on $E_{\mathbf{R}}$ with respect to the root α_j ($j = 1, \cdots, 6$). Then, from the definition,

$$g_1 = s_{12}, \quad g_2 = s_{123}, \quad g_3 = s_{23}, \quad g_4 = s_{34}, \quad g_5 = s_{45}, \quad g_6 = s_{56}.$$

Let E be the complexification of $E_{\mathbf{R}}$ and we extend the action of $W(E_6)$ on $E_{\mathbf{R}}$ to that on E in a natural manner. Moreover let $\mathbf{P}^5$ be the projective space associated to E. Then the $W(E_6)$-action on E induces a projective linear action of $W(E_6)$ on $\mathbf{P}^5$.

4. The configuration space of 6 points in $\mathbf{P}^2$

We have already defined a birational action of $W(E_6)$ on $\mathbf{C}^4$ in section 2. In this section, we explain that the birational transformations $s_1, \cdots, s_5, r$ naturally arise from the study of the configuration space of 6 points in $\mathbf{P}^2$.

For this purpose, we first introduce the linear space W of 3×6 matrices :

$$W = \{X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} \end{pmatrix}; \; x_{ij} \in \mathbf{C}\,(1 \leq i \leq 3,\ 1 \leq j \leq 6)\}.$$

Then W admits a left $GL(3,\mathbf{C})$-action and a right $GL(6,\mathbf{C})$-action in a natural way. For a moment, we identify $(\mathbf{C}^*)^6$ with the maximal torus of $GL(6,\mathbf{C})$ consisting of diagonal matrices and consider the action of $GL(3,\mathbf{C}) \times (\mathbf{C}^*)^6$ on W instead of that of $GL(3,\mathbf{C}) \times GL(6,\mathbf{C})$.

For simplicity, we write $X = (X_1, X_2)$ for the matrix $X \in W$, where both X_1, X_2 are 3×3 matrices. For any 3×3 matrix $Y = (y_{ij})_{1 \leq i,j \leq 3}$ with the condition $y_{ij} \neq 0$ $(1 \leq i,j \leq 3)$, we define a 3×3 matrix

$$\sigma(Y) = \left(\frac{1}{y_{ij}}\right)_{1 \leq i,j \leq 3}.$$

following a suggestion of M. Yoshida. Moreover, we put

$$D(i_1, i_2, i_3) = \det \begin{pmatrix} x_{1i_1} & x_{1i_2} & x_{1i_3} \\ x_{2i_1} & x_{2i_2} & x_{2i_3} \\ x_{3i_1} & x_{3i_2} & x_{3i_3} \end{pmatrix}$$

for a given matrix $X \in W$.

Using these notation, we define subsets W', W_0 of W by

$$W' = \{X \in W;\ D(i_1, i_2, i_3) \neq 0\,(1 \leq i_1 < i_2 < i_3 \leq 6)\},$$

$$W_0 = \{(X_1, X_2) \in W'; (I_3, Cof(X_1^{-1}X_2)),\ (I_3, \sigma(X_1^{-1}X_2)) \in W'\},$$

where $Cof(Y) = (\det Y)^t Y^{-1}$ is the cofactor matrix of a given square matrix Y.

It is clear that the action of $GL(3,\mathbf{C}) \times (\mathbf{C}^*)^6$ on W naturally induces that on each of W', W_0. In the sequel, we mainly consider the quotient space of W_0 unde the action of $GL(3,\mathbf{C}) \times (\mathbf{C}^*)^6$, that is,

$$W_Q = GL(3,\mathbf{C}) \setminus W_0/(\mathbf{C}^*)^6.$$

It is clear from the definition that for any element $X \in W_0$, there are $(g,h) \in GL(3,\mathbf{C}) \times (\mathbf{C}^*)^6$ and $(x_1, x_2, y_1, y_2) \in \mathbf{C}^4$ such that

$$gXh = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & x_1 & x_2 \\ 0 & 0 & 1 & 1 & y_1 & y_2 \end{pmatrix}.$$

In particular (x_1, x_2, y_1, y_2) is uniquely determined for $X \in W_0$. In this sense, $W_Q = GL(3,\mathbf{C}) \setminus W_0/(\mathbf{C}^*)^6$ is identified with an open subset of $\mathbf{C}^4$. Note that (x_1, x_2, y_1, y_2) is the variables (x_1, x_2, y_1, y_2) of section 2. Then $W_Q = \mathbf{C}^4 - T$.

Changes of column vectors of $X \in W_0$ induce birational transformations on $\mathbf{C}^4$ with coordinate system (x_1, x_2, y_1, y_2). The action s_j $(1 \leq j \leq 5)$ introduced in section 2 is nothing but the birational transformation on $\mathbf{C}^4$ corresponding to the change of the j-th column vector and $(j+1)$-column vector of $X \in W_0$. Moreover W_Q admits an involution induced from the action on W_0 defined by

$$\tilde{r} : (X_1, X_2) \longrightarrow (I_3, \sigma(X_1^{-1} X_2))$$

for any $(X_1, X_2) \in W_0$. The involution r defined in section 2 is equal to that induced from $\tilde{r}$.

The following theorem which seems known shows a concrete correspondence between $W(E_6)$ and the group $\tilde{G}$ introduced in section 2.

Theorem 4.1. The correspondence

$$g_1 \longrightarrow s_1, \quad g_2 \longrightarrow r, \quad g_3 \longrightarrow s_2, \quad g_4 \longrightarrow s_3, \quad g_5 \longrightarrow s_4, \quad g_6 \longrightarrow s_5$$

induces a group isomorphism of $W(E_6)$ to the group $\tilde{G}$.

Remark. In [3], it is stated that there is a $W(E_6)$-action on W_Q. See also [4].

5. $W(E_6)$-equivariant maps

We first define rational functions on E by

$$x_1(t) = \frac{h_{24} \cdot h_{234} \cdot h_{15} \cdot h_{135}}{h_{14} \cdot h_{134} \cdot h_{25} \cdot h_{235}}, \qquad x_2(t) = \frac{h_{24} \cdot h_{234} \cdot h_{16} \cdot h_{136}}{h_{14} \cdot h_{134} \cdot h_{26} \cdot h_{236}},$$

$$y_1(t) = \frac{h_{34} \cdot h_{234} \cdot h_{15} \cdot h_{125}}{h_{14} \cdot h_{124} \cdot h_{35} \cdot h_{235}}, \qquad y_2(t) = \frac{h_{34} \cdot h_{234} \cdot h_{16} \cdot h_{126}}{h_{14} \cdot h_{124} \cdot h_{36} \cdot h_{236}},$$

$$\lambda(t) = \frac{h_{34} \cdot h_{345} \cdot h_{26} \cdot h_{256}}{h_{24} \cdot h_{245} \cdot h_{36} \cdot h_{356}} \cdot \frac{h_{13} \cdot h_{136} \cdot h_{24} \cdot h_{246}}{h_{12} \cdot h_{126} \cdot h_{34} \cdot h_{346}},$$

$$\mu(t) = \frac{h_{456} \cdot h_{235} \cdot h_{134} \cdot h_{126}}{h \cdot h_{15} \cdot h_{24} \cdot h_{36}} \cdot \frac{h_{16} \cdot h_{136} \cdot h_{24} \cdot h_{234}}{h_{14} \cdot h_{134} \cdot h_{26} \cdot h_{236}},$$

$$\nu(t) = \frac{h_{25} \cdot h_{235} \cdot h_{46} \cdot h_{346}}{h_{24} \cdot h_{234} \cdot h_{56} \cdot h_{356}} \cdot \frac{h_{15} \cdot h_{156} \cdot h_{24} \cdot h_{246}}{h_{14} \cdot h_{146} \cdot h_{25} \cdot h_{256}},$$

$$\rho(t) = \frac{h_{24} \cdot h_{245} \cdot h_{36} \cdot h_{356}}{h_{23} \cdot h_{235} \cdot h_{46} \cdot h_{456}} \cdot \frac{h_{14} \cdot h_{146} \cdot h_{23} \cdot h_{236}}{h_{13} \cdot h_{136} \cdot h_{24} \cdot h_{246}},$$

where h, h_{ij}, h_{ijk} denote linear functions on E introduced in section 3. Since all the rational functions above are homogeneous of degree zero, they are regarded as functions on $\mathbf{P}^5$. Therefore defining

$$F_1(t) = (x_1(t), x_2(t), y_1(t), y_2(t)), \quad F_2(t) = (\lambda(t), \mu(t), \nu(t), \rho(t)),$$

we obtain two maps F_1, F_2 from $\mathbf{P}^5$ to $\mathbf{C}^4$. The roles of F_1, F_2 will become clear in Theorem 5.1 which will be given later. To define F_1, F_2, I am indebted to [4]. We are going to explain the meaning of $x_j(t)$, $y_j(t)$ following [4].

We begin with defining the cross ratio. Let $\xi_i = [\xi_{1i} : \xi_{2i} : \xi_{3i}]$ $(1 \leq i \leq 5)$ be five points of $\mathbf{P}^2$ and let $l : q_1u_1 + q_2u_2 + q_3u_3 = 0$ be a generic line in $\mathbf{P}^2$. We denote by $[1 : z_i : w_i]$ the intersection of l and the line passing through the points ξ_1 and ξ_i. Then we put

$$CR(\xi_2, \xi_3, \xi_4, \xi_5; \xi_1) = \frac{(z_2 - z_4)(z_3 - z_5)}{(z_2 - z_5)(z_3 - z_4)} \tag{1}$$

which is in fact a cross ratio of z_2, z_3, z_4, z_5.

Now we consider a matrix of the form

$$X = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & x_1 & x_2 \\ 0 & 0 & 1 & 1 & y_1 & y_2 \end{pmatrix}.$$

From the matrix X, we define six points $\xi_i\,(i = 1, \cdots, 6)$ in $\mathbf{P}^2$ in a usual manner, that is,

$$\xi_1 = [1 : 0 : 0], \quad \xi_2 = [0 : 1 : 0], \quad \xi_3 = [0 : 0 : 1],$$

$$\xi_4 = [1 : 1 : 1], \quad \xi_5 = [1 : x_1 : y_1], \quad \xi_6 = [1 : x_2 : y_2].$$

Then we can compute $CR(\xi_{i_2}, \xi_{i_3}, \xi_{i_4}, \xi_{i_5}; \xi_{i_1})$ explicitly for various i_1, i_2, i_3, i_4, i_5.

On the other hand, we put

$$CR'(i_2, i_3, i_4, i_5; i_1) = \frac{h_{i_2i_4}h_{i_1i_2i_4}h_{i_3i_5}h_{i_1i_3i_5}}{h_{i_3i_4}h_{i_1i_3i_4}h_{i_2i_5}h_{i_1i_2i_5}}. \tag{2}$$

By definition, $CR'(i_2, i_3, i_4, i_5; i_1)$ is a function on $\mathbf{P}^5$. Then from the equation

$$CR(\xi_{i_2}, \xi_{i_3}, \xi_{i_4}, \xi_{i_5}; \xi_{i_1}) = CR'(i_2, i_3, i_4, i_5; i_1), \tag{3}$$

we obtain various equalities. In particular, by computing the cases

$$(i_1, i_2, i_3, i_4, i_5) = (3, 2, 1, 4, 5), (3, 2, 1, 4, 6), (2, 1, 3, 4, 5), (2, 1, 3, 4, 6),$$

we have the definition of $x_1(t)$, $x_2(t)$, $y_1(t)$, $y_2(t)$ at the beginning of this section.

Let F_3 be the birational transformation on $\mathbf{C}^4$ defined by $F_3(x_1, x_2, y_1, y_2) = (\lambda, \mu, \nu, \rho)$, where

$$\lambda = \frac{x_2(x_1 - 1)(y_1 - y_2)(y_2 - 1)}{y_2(x_1 - x_2)(x_2 - 1)(y_1 - 1)},$$

$$\mu = \frac{\{(y_1 - 1)(x_2 - y_2) - (y_2 - 1)(x_1 - y_1)\}x_2y_2}{x_1x_2y_1 - x_1x_2y_2 - x_1y_1y_2 + x_1y_2 + x_2y_1y_2 - x_2y_1},$$

$$\nu = -\frac{(x_1y_2 - x_2y_1)(x_2 - 1)(y_2 - 1)}{(x_1 - x_2)(x_2 - y_2)(y_1 - y_2)},$$

$$\rho = \frac{(x_1 - x_2)(x_2 - y_2)(y_1 - 1)}{\{(x_1 - 1)(x_2 - y_2) - (x_1 - y_1)(x_2 - 1)\}(y_2 - 1)x_2}.$$

It is easy to show that F_3 is birational, because its inverse is given by

$$F_3^{-1}(\lambda,\mu,\nu,\rho)=\Big(\frac{(\lambda\rho-1)(\lambda\mu\nu\rho-1)}{(\lambda\mu\rho-1)(\lambda\nu\rho-1)},\frac{(\lambda\rho-1)\mu}{\lambda\mu\rho-1},\frac{(\mu\nu\rho-1)(\rho-1)}{(\mu\rho-1)(\nu\rho-1)},\frac{(\rho-1)\mu}{\mu\rho-1}\Big).$$

By the map F_3, the action of $W(E_6)$ on the (x_1,x_2,y_1,y_2)-space implies that on the (λ,μ,ν,ρ)-space. In fact, we define the following six birational transformations on the (λ,μ,ν,ρ)-space (cf.[7]):

$$\tilde{g}_1:\begin{cases}\lambda\longrightarrow & \lambda\mu\nu\rho^2(1-\lambda)/(\lambda\mu\nu\rho^2-1)\\ \mu\longrightarrow & (\lambda\mu\rho-1)(\lambda\mu\nu\rho-1)/(\mu(\lambda\rho-1)(\lambda\nu\rho-1))\\ \nu\longrightarrow & (\lambda\nu\rho-1)(\lambda\mu\nu\rho-1)/(\nu(\lambda\rho-1)(\lambda\mu\rho-1))\\ \rho\longrightarrow & (\lambda\rho-1)(\lambda\mu\nu\rho^2-1)/(\rho(\lambda-1)(\lambda\mu\nu\rho-1))\end{cases}$$

$$\tilde{g}_2:(\lambda,\mu,\nu,\rho)\longrightarrow(\lambda,1/\mu,\nu,\mu\rho)$$

$$\tilde{g}_3:(\lambda,\mu,\nu,\rho)\longrightarrow(1/\lambda,\mu,\nu,\lambda\rho)$$

$$\tilde{g}_4:(\lambda,\mu,\nu,\rho)\longrightarrow(\lambda\rho,\mu\rho,\nu\rho,1/\rho)$$

$$\tilde{g}_5:(\lambda,\mu,\nu,\rho)\longrightarrow(\lambda,\mu,1/\nu,\nu\rho)$$

$$\tilde{g}_6:\begin{cases}\lambda\longrightarrow & (\lambda\nu\rho-1)(\lambda\mu\nu\rho-1)/(\lambda(\nu\rho-1)(\mu\nu\rho-1))\\ \mu\longrightarrow & (\mu\nu\rho-1)(\lambda\mu\nu\rho-1)/(\mu(\nu\rho-1)(\lambda\nu\rho-1))\\ \nu\longrightarrow & \lambda\mu\nu\rho^2(1-\nu)/(\lambda\mu\nu\rho^2-1)\\ \rho\longrightarrow & (\nu\rho-1)(\lambda\mu\nu\rho^2-1)/(\rho(\nu-1)(\lambda\mu\nu\rho-1))\end{cases}$$

Let G_1 be the group generated by $\tilde{g}_j$ $(j=1,\cdots,6)$. Then the correspondence

$$g_j\longrightarrow\tilde{g}_j\quad j=1,\cdots,6$$

is an isomorphism between $W(E_6)$ and G_1.

Needless to say, F_1 (resp. F_2) is regarded as a map from $\mathbf{P}^5$ to the (x_1,x_2,y_1,y_2)-space (resp. the (λ,μ,ν,ρ)-space.) Moreover, F_3 is regarded as a map from the (x_1,x_2,y_1,y_2)-space to the (λ,μ,ν,ρ)-space.

Theorem 5.1. The three maps F_j $(j=1,2,3)$ are $W(E_6)$-equivariant and

$$F_3\circ F_1(g(t))=F_2(g(t))\quad(\forall t\in\mathbf{P}^5,\forall g\in W(E_6)).$$

The $W(E_6)$-equivariances of F_1, F_2 are stated in [4] implicitly.

We now mention the meaning of the (λ,μ,ν,ρ)-space. In [2], A. Cayley defined a 4-dimensional family of cubic surfaces. Modifying his family, we introduce a family of cubic surfaces of $\mathbf{P}^3$ with homogeneous coordinate $(X:Y:Z:W)$ depending on parameters (λ,μ,ν,ρ) as follows (cf. [7]):

$$\rho W[\lambda X^2+\mu Y^2+\nu Z^2+(\rho-1)^2(\lambda\mu\nu\rho-1)^2W^2$$

$$+(\mu\nu+1)YZ+(\lambda\nu+1)ZX+(\lambda\mu+1)XY$$

$$-(\rho-1)(\lambda\mu\nu\rho-1)W\{(\lambda+1)X+(\mu+1)Y+(\nu+1)Z\}]+XYZ=0.$$

The family of cubic surfaces above admits a $W(E_6)$-action as given in [7]. In particular, the $W(E_6)$-action in [7] preserves the parameter space. For this reason, we obtain a $W(E_6)$-action on the (λ,μ,ν,ρ)-space which actually coincides with the $W(E_6)$-action on the (λ,μ,ν,ρ)-space explained before Theorem 5.1.

6. A Conjecture of B. Hunt

It is known (cf.[1]) that there is a unique $W(E_6)$-invariant homogeneous polynomial of $t=(t_1,\cdots,t_6)$ of degree 5 up to a constant factor. For example, we take $Q_5(t)$ below as such a polynomial (cf. [4]):

$$Q_5(t)=-\frac{5}{108}t_6^5+\frac{5}{18}\sigma_1t_6^3+\frac{5}{4}(\sigma_1^2-4\sigma_2)t_6+30\sqrt{\sigma_5},$$

where $\sigma_i=\sigma_i(t_1^2,\cdots,t_5^2)$ is the i-th elementary symmetric polynomial in $t_1^2,\cdots,t_5^2$ and $\sqrt{\sigma_5}=t_1\cdots t_5$.

Let I_5 be the hypersurface in $\mathbf{P}^5$ defined by $Q_5(t)=0$. Since $Q_5(t)$ is $W(E_6)$-invariant, so is I_5. Moreover, since $\dim I_5=4$, the restrictions $F_1|I_5$, $F_2|I_5$ are generically finite maps from I_5 to $\mathbf{C}^4$. In [4], B. Hunt stated conjectures on these maps which turn out to be one conjecture below.

Conjecture 6.1.([4]) Both $F_1|I_5$, $F_2|I_5$ are generically bijective.

How to attack Conjecture 6.1 with the help of REDUCE? In virtue of Theorem 5.1, it suffices to show Conjecture 6.1 for one of $F_1|I_5$, $F_2|I_5$. Noting the definition of $F_1(t)$, we find that Conjecture 6.1 is rewritten as follows:

Problem 6.2. Let x_1,x_2,y_1,y_2 be constants. At least assume that (x_1,x_2,y_1,y_2) is outside the set T. Using x_1,x_2,y_1,y_2, we define four polynomials of t by

$$f_1=h_{24}\cdot h_{234}\cdot h_{15}\cdot h_{135}-x_1\cdot h_{14}\cdot h_{134}\cdot h_{25}\cdot h_{235},$$

$$f_2=h_{24}\cdot h_{234}\cdot h_{16}\cdot h_{136}-x_2\cdot h_{14}\cdot h_{134}\cdot h_{26}\cdot h_{236},$$

$$g_1=h_{34}\cdot h_{234}\cdot h_{15}\cdot h_{125}-y_1\cdot h_{14}\cdot h_{124}\cdot h_{35}\cdot h_{235},$$

$$g_2=h_{34}\cdot h_{234}\cdot h_{16}\cdot h_{126}-y_2\cdot h_{14}\cdot h_{124}\cdot h_{36}\cdot h_{236},$$

where h, h_{ij}, h_{ijk} are linear functions of t defined in section 3. Then how many solutions are there for the simultaneous equations of t defined by

$$f_1=f_2=g_1=g_2=Q_5=0 \tag{4}$$

under the condition $F_1(t)\notin T$?

Needles to say, there is a gap between Conjecture 6.1 and Problem 6.2, that is, Conjecture 6.1 claims that for generic x_1,x_2,y_1,y_2, equation (4) has a unique projective solution. Since I don't know whether Conjecture 6.1 is true or not, I reformulate it as a problem.

I tried to solve Problem 6.2 directly by using REDUCE3.4 on TOSHIBA J3100 once and at last abondaned to do because of out of capacity.

From now on, I am going to explain results related with Problem 6.2 and the moduli of cubic surfaces. We consider the hypersurface H in $\mathbf{P}^5$ defined by $\lambda(t)-1=0$, that is,

$$(5)\qquad P(t)=h_{345}\cdot h_{26}\cdot h_{256}\cdot h_{13}\cdot h_{136}\cdot h_{246}-h_{245}\cdot h_{36}\cdot h_{356}\cdot h_{12}\cdot h_{126}\cdot h_{346}=0.$$

Then it is easy to show that the polynomial $P(t)$ of equation (4) is decomposed into two factors (up to a constant):

$$P(t)=h_{23}\cdot P_5(t),$$

where $P_5(t)$ is homogeneous of degree 5. Moreover P_5 is so taken that

$$P_5(t_1,t_2,t_3,t_4,t_5,t_6)=const.Q_5(t_1,t_2,t_3,t_4,t_6,-3t_5).$$

From this remarkable relation, we easily imply the following (cf. [4], [6]).

Proposition 6.3.

(i) There are 45 hypersurfaces in $\mathbf{P}^5$ as the $W(E_6)$-orbit of H. Moreover, the isotropy subgroup of H in $W(E_6)$ is isomorphic to the Weyl group of type F_4.

(ii) The intersection $H\cap I_5$ is decomposed into two irreducible components. One is defined by $t_5=t_6=0$ therefore is isomorphic to $\mathbf{P}^3$. The other is defined by an equation of degree 24.

(iii) If $t\in H$, then $F_2(t)=(1,1,1,1)$, that is, $\lambda(t)=\mu(t)=\nu(t)=\rho(t)=1$. The corresponding cubic surface has Eckard points.

It follows from Proposition 6.3 (i) that there is a natural 1-1 correspondence between the $W(E_6)$-orbit of H and the 45 exceptional divisors of Naruki's cross ratio variety [6].

We mention Proposition 6.3 (ii) in detail. We first introduce symmetric polynomials of t_1,t_2,t_3,t_4 by

$$s_2=t_1^2+t_2^2+t_3^2+t_4^2,$$
$$s_4=t_1^2(t_2^2+t_3^2+t_4^2)+t_2^2(t_3^2+t_4^2)+t_3^2t_4^2,$$
$$s_4'=t_1t_2t_3t_4.$$

Using s_2, s_4, s_4', we define the polynomial h of degree 24 by

$$h=c_{10}t_5^{20}+c_9t_5^{18}+c_8t_5^{16}+c_7t_5^{14}+c_6t_5^{12}+c_5t_5^{10}+c_4t_5^8+c_3t_5^6+c_2t_5^4+c_1t_5^2+c_0,$$

where

$$c_{10}=1728s_2^2,$$
$$c_9=432s_2(-21s_2^2+20s_4),$$
$$c_8=27(4800s_4'^2+761s_2^4-1736s_2^2s_4+400s_4^2),$$
$$c_7=8s_2(-46656s_4'^2-3217s_2^4+12852s_2^2s_4-10368s_4^2),$$

$$c_6 = 2(-190080s_4'^2 s_2^2 - 336960s_4'^2 s_4 + 9251s_2^6 - 55955s_2^4 s_4$$
$$+91368s_2^2 s_4^2 - 28080s_4^3),$$
$$c_5 = 2s_2(825360s_4'^2 s_2^2 - 1582848s_4'^2 s_4 - 3256s_2^6 + 27143s_2^4 s_4$$
$$-72496s_2^2 s_4^2 + 61776s_4^3),$$
$$c_4 = -59833728s_4'^4 - 1370994s_4'^2 s_2^4 + 5809680s_4'^2 s_2^2 s_4 - 4732128s_4'^2 s_4^2 - 193s_2^8$$
$$+3054s_2^6 s_4 - 12981s_2^4 s_4^2 + 10120s_2^2 s_4^3 + 21168s_4^4,$$
$$c_3 = 2s_2(-2191104s_4'^4 + 199476s_4'^2 s_2^4 - 1263024s_4'^2 s_2^2 s_4 + 1990080s_4'^2 s_4^2$$
$$+496s_2^8 - 7327s_2^6 s_4 + 40443s_2^4 s_4^2 - 98824s_2^2 s_4^3 + 90160s_4^4),$$
$$c_2 = -907200s_4'^4 s_2^2 + 2491776s_4'^4 s_4 - 54714s_4'^2 s_2^6 + 554274s_4'^2 s_2^4 s_4$$
$$-1854576s_4'^2 s_2^2 s_4^2 + 2051616s_4'^2 s_4^3 - 256s_2^10 + 4640s_2^8 s_4 - 33505s_2^6 s_4^2$$
$$+120460s_2^4 s_4^3 - 215600s_2^2 s_4^4 + 153664s_4^5,$$
$$c_1 = 6s_4'^2 s_2(-4968s_4'^2 s_2^2 + 14688s_4'^2 s_4 - 26s_2^6 + 285s_2^4 s_4 - 1032s_2^2 s_4^2 + 1232s_4^3),$$
$$c_0 = 27s_4'^4(192s_4'^2 + s_2^4 - 8s_2^2 s_4 + 16s_4^2).$$

Moreover,

$$N = -2\{(5s_2^5 - 1602s_2^4 t_5^2 - 34s_2^3 s_4 + 4134s_2^3 t_5^4 + 10037s_2^2 s_4 t_5^2 - 3005s_2^2 t_5^6$$
$$+56s_2 s_4^2 - 12820s_2 s_4 t_5^4 + 828s_2 t_5^8 - 15764s_4^2 t_5^2 + 1980s_4 t_5^6 - 360t_5^{10})t_5^2$$
$$-(s_2^2 + 164s_2 t_5^2 - 4s_4 + 7368t_5^4)s_4'^2\}s_4' t_5,$$
$$D = -\{3(31s_2^3 + 650s_2^2 t_5^2 - 92s_2 s_4 + 2320s_2 t_5^4 - 1752s_4 t_5^2 + 5648t_5^6)s_4'^2 t_5^2$$
$$+2(2464s_4^2 - 2055s_4 t_5^4 + 187t_5^8)s_2^2 t_5^4 - 4(1687s_4^2 - 415s_4 t_5^4 + 12t_5^8)s_2 t_5^6$$
$$-(1465s_4 - 1044t_5^4)s_2^4 t_5^4 + 15(269s_4 - 61t_5^4)s_2^3 t_5^6 - 16s_4'^4 + 144s_2^6 t_5^4$$
$$-599s_2^5 t_5^6 - 5488s_4^3 t_5^4 + 2072s_4^2 t_5^8 - 120s_4 t_5^{12}\}.$$

Then from the equations

$$P_5 = Q_5 = 0,$$

we obtain

$$t_6 = N/D, \quad h = 0.$$

The equation $h = 0$ is the one stated in Proposition 6.3 (ii).

If we consider the equation $\lambda - 1 = 0$ in the (x_1, x_2, y_1, y_2)-space, we obtain a hypersurface H_0 defined by

$$x_2(x_1 - 1)(y_1 - y_2)(y_2 - 1) - y_2(x_1 - x_2)(x_2 - 1)(y_1 - 1) = 0. \tag{6}$$

Now we formulate a problem simplified from Problem 6.2, noting Proposition 6.3 (ii). Namely, we consider Problem 6.2 in the case $t_5 = t_6 = 0$ and $t_1 = 1$. (The condition $t_1 = 1$ is not essential. From the homogeneity, we may assume $t_j = 1$ for some j.)

Problem 6.2'. Define four polynomials of t_2, t_3, t_4 by

$$f_{10} = (t_2+t_3-t_4+1)^2(t_2+t_4)(t_3-1) - x_1(t_2+t_3)(t_2-t_3+t_4+1)^2(t_4-1),$$

$$\begin{aligned} f_{20} &= (t_2+t_3+t_4+1)(t_2+t_3-t_4+1)(t_3-1)t_2 \\ &\quad + x_2(t_2+t_3)(t_2-t_3+t_4+1)(t_2-t_3-t_4+1), \end{aligned}$$

$$g_{10} = (t_2+t_3-t_4+1)^2(t_2-t_3)(t_4+1) - y_1(t_2-t_3+t_4+1)^2(t_2-t_4)(t_3+1),$$

$$\begin{aligned} g_{20} &= (t_2+t_3+t_4+1)(t_2+t_3-t_4+1)(t_2-t_3) \\ &\quad - y_2(t_2-t_3+t_4+1)(t_2-t_3-t_4+1)(t_3+1)t_2, \end{aligned}$$

where x_1, x_2, y_1, y_2 are constants with the condition (6) and $(x_1, x_2, y_1, y_2) \notin T$. (In particular, we assume that x_1 is a rational function of x_2, y_1, y_2.) Then how many solutions are there for the equations (7) of t_2, t_3, t_4 below

$$f_{10} = f_{20} = g_{10} = g_{20} = 0 \tag{7}$$

under the condition $t \notin T$?

It is possible to give an answer to Problem 6.2'. In fact, erasing t_3, t_4 from (7), we obtain an equation for t_2 defined by

$$\sum_{j=0}^{9} b_j t_2^j = 0, \tag{8}$$

where

$$b_9 = (x_2y_1y_2 - x_2y_2 - 2y_1y_2 + y_1 + y_2^2)^2(x_2y_2 - 2y_2 + 1)y_2^4,$$

$$b_8 = 3(x_2y_1y_2 - x_2y_2 - 2y_1y_2 + y_1 + y_2^2)^2(x_2y_2 - 2x_2 + 1)y_2^4,$$

$$\begin{aligned} b_6 = -4&(x_2^2y_1y_2 - x_2^2y_2 + x_2y_1^2 + x_2y_1y_2^2 - 4x_2y_1y_2 + x_2y_1 + x_2y_2^2 - y_1^2y_2 + y_1y_2) \\ &\times(x_2y_1y_2 - x_2y_2 - 2y_1y_2 + y_1 + y_2^2)(x_2y_2 - 2x_2 + 1)y_2^3, \end{aligned}$$

$$\begin{aligned} b_5 = -6&(x_2y_1y_2 - x_2y_2 - 2y_1y_2 + y_1 + y_2^2)(x_2y_1 + x_2y_2^2 - 2x_2y_2 - y_1y_2 + y_2) \\ &\times(x_2y_2 - 2y_2 + 1)x_2y_1y_2^2, \end{aligned}$$

$$\begin{aligned} b_4 = 6&(x_2y_1y_2 - x_2y_2 - 2y_1y_2 + y_1 + y_2^2)(x_2y_1 + x_2y_2^2 - 2x_2y_2 - y_1y_2 + y_2) \\ &\times(x_2y_2 - 2x_2 + 1)x_2y_1y_2^2, \end{aligned}$$

$$\begin{aligned} b_3 = 4&(x_2^2y_1y_2 - x_2^2y_2 + x_2y_1^2 + x_2y_1y_2^2 - 4x_2y_1y_2 + x_2y_1 + x_2y_2^2 - y_1^2y_2 + y_1y_2) \\ &\times(x_2y_1 + x_2y_2^2 - 2x_2y_2 - y_1y_2 + y_2)(x_2y_2 - 2y_2 + 1)x_2y_1y_2, \end{aligned}$$

$$b_1 = -3(x_2y_1 + x_2y_2^2 - 2x_2y_2 - y_1y_2 + y_2)^2(x_2y_2 - 2y_2 + 1)x_2^2y_1^2,$$

$$b_0 = -(x_2y_1 + x_2y_2^2 - 2x_2y_2 - y_1y_2 + y_2)^2(x_2y_2 - 2x_2 + 1)x_2^2y_1^2,$$
$$b_7 = b_2 = 0.$$

Moreover, if t_2 is a solution of (8), t_3, t_4 are uniquely determined by (7).

I checked that equation (8) for t_2 is irreducible of degree 9 and that for generic x_2, y_1, y_2, (8) has no multiple factor. As a consequence, we obtain the following.

Theorem 6.4. The restriction of F_1 to the subspace $t_5 = t_6 = 0$ is generically 9 to 1.

I am not sure whether Theorem 6.4 induces the invalidity of Conjecture 6.1 or not.

Acknowledgements

Last I mention that the note [4] and the communications with Prof. B. Hunt are valuable when I formulate the maps F_j $(j = 1, 2, 3)$ and solve Problem 6.2'. Moreover, I am indebted to Prof. K. Okubo because without his help, I could not use REDUCE 3.4 (not REDUCE 3.2 !) which is a powerful tool in obtaining the results of this note.

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